A Reinvestigation of the Problem of the Automorphic Linear Transformation of a Bipartite Quadric.

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1. If the bipartite quadric be

$$\begin{array}{cccc|c} x & y & z & \cdots\cdots & \\ \hline a_1 & a_2 & a_3 & \cdots\cdots & x' \\ b_1 & b_2 & b_3 & \cdots\cdots & y' \\ c_1 & c_2 & c_3 & \cdots\cdots & z' \\ \cdots & \cdots & \cdots & \cdots\cdots & \cdots \end{array}$$

the problem is to find two matrices

$$\left(\begin{array}{cccc} \alpha_1 & \alpha_2 & \alpha_3 & \cdots\cdots \\ \beta_1 & \beta_2 & \beta_3 & \cdots\cdots \\ \gamma_1 & \gamma_2 & \gamma_3 & \cdots\cdots \\ \cdots & \cdots & \cdots & \cdots\cdots \end{array}\right), \qquad \left(\begin{array}{cccc} m_1 & m_2 & m_3 & \cdots\cdots \\ n_1 & n_2 & n_3 & \cdots\cdots \\ r_1 & r_2 & r_3 & \cdots\cdots \\ \cdots & \cdots & \cdots & \cdots\cdots \end{array}\right)$$

such that after performing the substitutions

$$(x,\ y,\ z,\ \ldots\ldots) = \left(\begin{array}{cccc} \alpha_1 & \alpha_2 & \alpha_3 & \cdots\cdots \\ \beta_1 & \beta_2 & \beta_3 & \cdots\cdots \\ \gamma_1 & \gamma_2 & \gamma_3 & \cdots\cdots \\ \cdots & \cdots & \cdots & \cdots\cdots \end{array}\right)(\xi,\ \eta,\ \zeta\ \ldots\ldots),$$

$$(x',\ y',\ z',\ \ldots\ldots) = \left(\begin{array}{cccc} m_1 & m_2 & m_3 & \cdots\cdots \\ n_1 & n_2 & n_3 & \cdots\cdots \\ r_1 & r_2 & r_3 & \cdots\cdots \\ \cdots & \cdots & \cdots & \cdots\cdots \end{array}\right)(\xi',\ \eta',\ \zeta',\ \ldots\ldots),$$

we shall have

$$\begin{array}{l} \quad x, \quad y, \quad z, \ \ldots\ldots \\ \hline \begin{array}{cccc|c} a_1 & a_2 & a_3 & \ldots\ldots & x' \\ b_1 & b_2 & b_3 & \ldots\ldots & y' \\ c_1 & c_2 & c_3 & \ldots\ldots & z' \\ \ldots & \ldots & \ldots & \ldots\ldots & \ldots \end{array} \end{array} = \begin{array}{l} \quad \xi, \quad \eta, \quad \zeta, \ \ldots\ldots \\ \hline \begin{array}{cccc|c} a_1 & a_2 & a_3 & \ldots\ldots & \xi' \\ b_1 & b_2 & b_3 & \ldots\ldots & \eta' \\ c_1 & c_2 & c_3 & \ldots\ldots & \zeta' \\ \ldots & \ldots & \ldots & \ldots\ldots & \ldots \end{array} \end{array}$$

2. The mere performance of the substitutions changes the given bipartite quadric into

$$\begin{array}{l} \quad \dfrac{\alpha_1 \ \alpha_2 \ \alpha_3 \ldots\ldots}{\xi \ \ \eta \ \ \zeta \ \ldots\ldots} \qquad \dfrac{\beta_1 \ \beta_2 \ \beta_3 \ldots\ldots}{\xi \ \ \eta \ \ \zeta \ \ldots\ldots} \qquad \dfrac{\gamma_1 \ \gamma_2 \ \gamma_3 \ldots\ldots}{\xi \ \ \eta \ \ \zeta \ \ldots\ldots} \\ \hline \begin{array}{cccc|c} a_1 & a_2 & a_3 & \ldots\ldots & \dfrac{m_1 \ m_2 \ m_3 \ldots\ldots}{\xi' \ \ \eta' \ \ \zeta' \ \ldots\ldots} \\ b_1 & b_2 & b_3 & \ldots\ldots & \dfrac{n_1 \ n_2 \ n_3 \ldots\ldots}{\xi' \ \ \eta' \ \ \zeta' \ \ldots\ldots} \\ c_1 & c_2 & c_3 & \ldots\ldots & \dfrac{r_1 \ r_2 \ r_3 \ldots\ldots}{\xi' \ \ \eta' \ \ \zeta' \ \ldots\ldots} \end{array} \end{array}$$

which by a property of bipartite functions,*

$$= \begin{array}{l} \quad \xi \quad \eta \quad \zeta \ \ldots\ldots \\ \hline \begin{array}{cccc|cccc} \alpha_1 & \alpha_2 & \alpha_3 & \ldots\ldots & a_1 & b_1 & c_1 & \ldots\ldots \\ \beta_1 & \beta_2 & \beta_3 & \ldots\ldots & a_2 & b_2 & c_2 & \ldots\ldots \\ \gamma_1 & \gamma_2 & \gamma_3 & \ldots\ldots & a_3 & b_3 & c_3 & \ldots\ldots \\ \ldots & \ldots & \ldots & \ldots\ldots & \ldots & \ldots & \ldots & \ldots\ldots \end{array} \end{array} \begin{array}{l} \\ \\ \\ \\ \hline \begin{array}{cccc|c} m_1 & n_1 & r_1 & \ldots\ldots & \xi' \\ m_2 & n_2 & r_2 & \ldots\ldots & \eta' \\ m_3 & n_3 & r_3 & \ldots\ldots & \zeta'. \\ \ldots & \ldots & \ldots & \ldots\ldots & \ldots \end{array} \end{array}$$

Consequently if we denote the square array of the original quadric by D, and

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those of the two substitutions by S_1, S_2 respectively, the transformed function may be written*

$$\begin{array}{l|l|l}
\xi \quad \eta \quad \zeta \ldots\ldots & & \\
\hline
\qquad S_1 & tr\, D & \\
\cline{2-2}
 & tr\, S_2 & \xi' \\
 & & \eta' \\
 & & \zeta'. \\
 & & \vdots
\end{array}$$

But this by another property of bipartites is equal to

$$\begin{array}{l|l}
\xi \quad \eta \quad \zeta \ldots\ldots & \\
\hline
tr\, S_2 \, . \, D \, . \, S_1 & \xi' \\
 & \eta' \\
 & \zeta',
\end{array}$$

so that the problem is changed into finding two matrices S_1 and S_2 which will satisfy the equation

$$tr\, S_2 \, . \, D \, . \, S_1 = D.$$

The problem is evidently indeterminate, there being two unknown matrices and one matrical equation to be satisfied, or $2n^2$ scalar unknowns and n^2 scalar equations for obtaining them; the solution therefore should involve n^2 arbitrary scalar constants.

3. It is immediately clear that we shall obtain a solution—and the most natural one—by putting S_1 or S_2 equal to any arbitrary matrix and then solving for S_2 or S_1. Doing this and denoting the arbitrary matrix by M we have the alternative solutions

$$\left.\begin{aligned} S_1 &= M, \\ tr\, S_2 &= DM^{-1}D^{-1}, \end{aligned}\right\}$$

$$\left.\begin{aligned} tr\, S_2 &= M, \\ S_1 &= D^{-1}M^{-1}D. \end{aligned}\right\}$$

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* "tr D" is used, as by Cayley, for the 'transverse' of D, the symbol tr indicating the operation of row-and-column exchange: "conj. D" would have been more appropriate, considering the way in which 'conjugate' is already used in connection with the elements, terms and minors of a determinant.

4. A symmetry more in keeping with the character of the equation will be attained by putting S_1 equal to the product of an arbitrary matrix and D, and then as before solving for S_2. This course of procedure gives us

$$\left.\begin{aligned} S_1 &= MD, \\ tr\, S_2 &= M^{-1}D^{-1}, \end{aligned}\right\}$$

or

$$\left.\begin{aligned} S_1 &= M^{-1}D, \\ tr\, S_2 &= MD^{-1}. \end{aligned}\right\}$$

In all these forms of solution, it will be observed, M, though in other respects arbitrary, must not have a vanishing determinant.

5. Taking one of the forms, say the third, we have the following simple theorem for the transformation in question:

The transformation of

$$(x, y, z, \ldots \between D \between x', y', z', \ldots) \text{ into } (\xi, \eta, \zeta, \ldots \between D \between \xi', \eta', \zeta' \ldots)$$

can be effected by the substitutions

$$\begin{aligned} (x, y, z, \ldots) &= \ (MD \quad \between \xi, \eta, \zeta, \ldots), \\ (x', y', z', \ldots) &= tr\,(M^{-1}D^{-1} \between \xi', \eta', \zeta', \ldots), \end{aligned}$$

where M is any arbitrary matrix with a non-vanishing determinant.

6. For the case of an ordinary quadric, viz. where D is axisymmetric, equal to A say, and $S_2 = S_1$, $= S$ say, the equation for solution is

$$tr\, S . A . S = A.$$

Now the left-hand member, as well as the right-hand member, of this is axisymmetric; for by the law of row-and-column exchange in a product

$$\begin{aligned} tr\,(tr\, S . A . S) &= tr\, S . tr\, A . tr\,(tr\, S) \\ &= tr\, S . A . S. \end{aligned}$$

The state of matters here therefore is that though there is one unknown matrix and one matrical equation to determine it, there are n^2 scalar unknowns, and only $\frac{1}{2}n(n+1)$ scalar equations to be satisfied. The problem is thus again an indeterminate one, there being $\frac{1}{2}n(n-1)$ arbitrary constants to be expected in the solution.

One solution of the equation is evident, viz.

$$tr\, S = -1;$$

it follows therefore that $tr\,S+1$ is a factor of $tr\,S.A.S-A$. Denoting the other factors—the first terms of which must evidently be A and S—by

$$A+x \text{ and } S+y,$$

and bearing in mind that the product of the three is axisymmetric, we see from the law of row-and-column exchange in a product that

$$y=1$$

and $\qquad x$ is axisymmetric.

The first and third factors being thus known, the second is readily found to be

$$A-A(1-S)^{-1}-(1-tr\,S)^{-1}A$$

and our equation is

$$(tr\,S+1)\{A-A(S+1)^{-1}-(tr\,S+1)^{-1}A\}(S+1)=0.$$

The solution got from the first factor is clearly identical with that got from the third. Taking the remaining factor we have, on doubling it,

$$A-2A(S+1)^{-1}+A-(tr\,S+1)^{-1}A=0$$

or

$$\{A-2A(S+1)^{-1}\}+tr\{A-2A(S+1)^{-1}\}=0,$$

the solution of which is evidently

$$A-2A(S+1)^{-1}=\text{any zero-axial skew matrix.}$$

Denoting by $-N$ such a partially arbitrary matrix—that is to say, a matrix involving as was desired $\frac{1}{2}n(n-1)$ different elements—we have

$$2A(S+1)^{-1}=A+N,$$

and $\therefore$

$$\tfrac{1}{2}(S+1)A^{-1}=(A+N)^{-1},$$

and finally

$$S=2(A+N)^{-1}A-1.$$

7. We have thus reached the following theorem:

The transformation of

$$(x,y,z,\ldots\ldots)(A)(x,y,z,\ldots\ldots) \text{ into } (\xi,\eta,\zeta,\ldots\ldots)(A)(\xi,\eta,\zeta)$$

can be effected by the substitution

$$(x,y,z,\ldots\ldots)=(2(A+N)^{-1}A-1)(\xi,\eta,\zeta,\ldots\ldots),$$

where N is any zero-axial skew matrix arbitrarily chosen except for the condition that the determinant of the matrix $A+N$ must not be zero.

Or, following Cayley's example when dealing with the much simpler case of an orthogonal substitution, we may put the result in the form of a "rule," viz.

To obtain the coefficients α_{rs} of the linear substitution which transforms the quadric

$$ax^2+by^2+cz^2+\ldots\ldots+2fyz+2gzx+2hxy+\ldots\ldots$$

into itself, form the determinant K adjugate to

$$\left|\begin{array}{cccc} a & h+\nu & g-\mu & \ldots\ldots \\ h-\nu & b & f+\lambda & \ldots\ldots \\ g+\mu & f-\lambda & c & \ldots\ldots \\ \ldots\ldots & \ldots\ldots & \ldots\ldots & \ldots\ldots \end{array}\right| \text{ or } \Delta$$

and then put

$$\Delta\,.\,\alpha_{rs}=2\,(r^{th}\ col.\ of\ K \between s^{th}\ col.\ of\ A) \qquad \text{when } r\neq s$$

and $$\Delta\,.\,\alpha_{rr}=2\,(r^{th}\ col.\ of\ K \between r^{th}\ col.\ of\ A)-\Delta,$$

A being the discriminant of the quadric.

This is manifestly just as concise as Cayley's rule for the very simplest case. It is seen to pass into the latter when $a=b=c=\ldots\ldots=1$ and $f=g=h=\ldots\ldots=0$.

8. It should be noticed that the equation

$$tr\,S\,.\,A\,.\,S=A$$

resembles an ordinary algebraical equation which has only even powers of the unknown; for, if $S=\mu$ be a solution, it is clear that $S=-\mu$ is also a solution. Indeed instead of resolving $tr\,S\,.\,A\,.\,S-A$ into

$$(tr\,S+1)\{A-A\,(S+1)^{-1}-(tr\,S+1)^{-1}A\}(S+1),$$

we might have resolved it into

$$(tr\,S-1)\{A-A\,(1-S)^{-1}-(1-tr\,S)^{-1}A\}(S-1),$$

the solution $tr\,S=1$ being, to say the least, just as manifest as the solution $tr\,S=-1$.

The roots obtained are thus:

$$\pm 1,\ \pm\{2\,(A+N)^{-1}A-1\}.$$

9. Another point worthy of attention is the fact that the identity of

$$(tr\, S+1)\{A-A(S+1)^{-1}-(tr\, S+1)^{-1}A\}(S+1)$$

and $$tr\, S.A.S-A$$

is quite independent of the form of A, and that therefore whatever A may be, the root of

$$A-A(S+1)^{-1}-(tr\, S+1)^{-1}A=0$$

is a root of

$$tr\, S.A.S-A=0.$$

When, however, we proceeded, towards the close of §6, to solve for the root of the former equation, we introduced the condition $A=tr\, A$: consequently when A is unconditioned* it cannot be expected that the root thus reached will be a root of the equation

$$tr\, S.A.S-A=0.$$

10. Returning now to the general equation with which we started, viz.

$$tr\, S_2.D.S_1=D,$$

let us give S_1 a form analogous to the form $2(A+N)^{-1}A-1$ obtained for S in §6, changing A, of course, into the more general D and, in order that we may have the full number of arbitrary constants, putting a perfectly arbitrary matrix M in place of the zero-axial skew matrix N. Our equation then becomes

$$tr\, S_2.D.\{2(D+M)^{-1}D-1\}=D$$

or $$tr\, S_2.\{2D(D+M)^{-1}D-D\}=D$$

or $$tr\, S_2.\{2D(D+M)^{-1}-1\}D=D,$$

and ∴ $$tr\, S_2=\{2D(D+M)^{-1}-1\}^{-1}.$$

11. In place therefore of the simple theorem of §5 we have the following as a useful alternative:

The transformation of the bipartite quadric

$$(x,y,z,\ldots)(D)(x',y',z',\ldots) \text{ into } (\xi,\eta,\zeta,\ldots)(D)(\xi',\eta',\zeta',\ldots)$$

* The solution of $A-A(S+1)^{-1}-(tr\, S+1)^{-1}A=0$, when A is unconditioned, does not seem to be easy. As we see readily from the complete equation $tr\, S.A.S=A$, there are then implied n^2 equations for the determination of n^2 unknowns.

or, as the saying is, into itself, can be effected by the substitutions

$$(x, y, z, \ldots\ldots) = \quad (2\Delta^{-1}D - 1)(\xi, \eta, \zeta, \ldots\ldots),$$
$$(x', y', z', \ldots\ldots) = tr\,(2D\Delta^{-1} - 1)(\xi', \eta', \zeta', \ldots\ldots),$$

where $\Delta = D +$ *an arbitrary matrix and has a non-vanishing determinant.*

12. Bearing in mind the equation $tr\,S_2 . D . S_1 = D$, we see that the foregoing result rests finally on the identity

$$(2D\Delta^{-1} - 1)^{-1}D(2\Delta^{-1}D - 1) = D,$$

which is the same as

$$D(2\Delta^{-1}D - 1) = (2D\Delta^{-1} - 1)D.^{*}$$

This latter however leads with equal naturalness to the identity

$$(2D\Delta^{-1} - 1)D(2\Delta^{-1}D - 1)^{-1} = D,$$

so that another form of the solution of the equation

$$tr\,S_2 . D . S_1 = D$$

is available, viz.

$$\left.\begin{array}{l} S_1 = (2\Delta^{-1}D - 1)^{-1}, \\ S_2 = tr\,(2D\Delta^{-1} - 1). \end{array}\right\}$$

13. The existence of an alternative form of solution is, of course, what might have been expected from the character of the function under discussion, which itself has two forms:

$$\begin{array}{cccc|c} x & y & z & \ldots\ldots & \\ \hline a_1 & a_2 & a_3 & \ldots\ldots & x' \\ b_1 & b_2 & b_3 & \ldots\ldots & y' \\ c_1 & c_2 & c_3 & \ldots\ldots & z' \\ \ldots & \ldots & \ldots & \ldots\ldots & \ldots \end{array} \quad \text{and} \quad \begin{array}{cccc|c} x' & y' & z' & \ldots\ldots & \\ \hline a_1 & b_1 & c_1 & \ldots\ldots & x \\ a_2 & b_2 & c_2 & \ldots\ldots & y \\ a_3 & b_3 & c_3 & \ldots\ldots & z \\ \ldots & \ldots & \ldots & \ldots\ldots & \ldots \end{array}$$

or $$(x \quad y \quad z \ldots\ldots)(D)(x' \quad y' \quad z' \ldots\ldots),$$
and $$(x' \quad y' \quad z' \ldots\ldots)(tr\,D)(x \quad y \quad z \ldots\ldots).$$

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* The ultimate basis is, of course, the fact that

$$2D\Delta^{-1}D - D$$

is equal to either $$D(2\Delta^{-1}D - 1) \quad \text{or} \quad (2D\Delta^{-1} - 1)D.$$

Consequently, when we obtained the solution

$$\left.\begin{aligned} S_1 &= 2\Delta^{-1}D - 1, \\ S_2 &= tr\,(2D\Delta^{-1} - 1)^{-1}, \end{aligned}\right\}$$

we might, by attending to the evidently permissible interchange, have foretold the solution

$$\left.\begin{aligned} S_2 &= 2\,tr\,\Delta^{-1}\,tr\,D - 1, \\ S_1 &= tr\,(2\,tr\,D\,tr\,\Delta^{-1} - 1)^{-1}. \end{aligned}\right\}$$

And this latter will be found to be identical with the second solution obtained above; for—to take only the value for S_1—by the laws of matrix algebra

$$\begin{aligned} tr\,(2\,tr\,D\,tr\,\Delta^{-1} - 1)^{-1} &= \{tr\,(2\,tr\,D\,tr\,\Delta^{-1} - 1)\}^{-1} \\ &= \{tr\,(2\,tr\,D\,tr\,\Delta^{-1}) - 1\}^{-1} \\ &= (2\Delta^{-1}D - 1)^{-1}. \end{aligned}$$

14. Further, however, it must be noted that there is no reason a priori for the unlikeness—which is notable in both of these solutions—between the value of S_1 and the value of S_2; and the presumption to which we are thus led that either form must be suitable for both, is fully borne out on investigation. Thus, taking the form for $tr\,S_2$ in the first solution, we have

$$\begin{aligned} (2D\Delta^{-1} - 1)^{-1} &= \{(2D - \Delta)\,\Delta^{-1}\}^{-1}, \\ &= \Delta\,(2D - \Delta)^{-1}, \\ &= \{2D - (D - M)\}(D - M)^{-1}, \\ &= 2D\,(D - M)^{-1} - 1, \end{aligned}$$

which closely resembles the form for S_1 in the same solution. Similarly, taking the form for S_1, we have

$$\begin{aligned} 2\Delta^{-1}D - 1 &= \Delta^{-1}(2D - M), \\ &= \Delta^{-1}(D - M), \\ &= \{(D - M)^{-1}\Delta\}^{-1}, \\ &= \{(D - M)^{-1}(2D - D - M)\}^{-1}, \\ &= \{2D\,(D - M)^{-1} - 1\}^{-1}, \end{aligned}$$

which is quite similar to the form for $tr\,S_2$ in the same solution.

15. And this is not all, for, since

$$(D + M)^{-1}(D + M) = 1,$$

and $\therefore$ $$(D + M)^{-1}D + (D + M)^{-1}M = 1,$$

it follows that

$$2(D+M)^{-1}D-1 \text{ or } S_1=1-2(D+M)^{-1}M,$$

and similarly,

$$2D(D-M)^{-1}-1 \text{ or } tr\, S_2=1+2M(D-M)^{-1}.$$

It also appears incidentally in the preceding paragraph that

$$\left.\begin{array}{l} S_1=(D+M)^{-1}(D-M)\quad , \\ tr\, S_2=(D+M)\quad (D-M)^{-1}. \end{array}\right\}$$

and

16. We have therefore four identical expressions for S_1 in our first solution, and four identical expressions for S_2, each of the former being quite similar in form to one of the latter, viz.

$$\begin{array}{lll} S_1=2(D+M)^{-1}D-1, & \text{and} & tr\, S_2=2D(D-M)^{-1}-1, \\ \quad =1-2(D+M)^{-1}M, & & \quad =1+2M(D-M)^{-1}, \\ \quad =(D+M)^{-1}(D-M), & & \quad =(D+M)(D-M)^{-1}, \\ \quad =\{2(D-M)^{-1}D-1\}^{-1}, & & \quad =\{2D(D+M)^{-1}-1\}^{-1}. \end{array}$$

Looking at the first and last of the four forms we see that each of the eight expressions is such that to change the sign of M in it is the same as to take its reciprocal. Since therefore, in the second solution of §13, the values of S_1 and S_2 are respectively the reciprocals of S_1 and S_2 in the first solution, it follows that the second solution can be got from the first by merely changing the sign of M. But M is entirely arbitrary; consequently the second solution is not really different from the first.

17. The only advantage of these more complicated forms of S_1 and S_2 is that specialization from them is very easy.

Thus, when D is axisymmetric, equal to A say, and $S_1 = S_2$, we have for the conditioning of M

$$2(A+M)^{-1}A-1=tr\{2A(A-M)^{-1}-1\},$$

whence

$$(A+M)^{-1}A=tr\,(A-M)^{-1}\,tr\,A,$$

and ∴

$$(A+M)^{-1}=(tr\,A-tr\,M)^{-1},$$

and finally,

$$M=-tr\,M,$$

which is the definition of a zero-axial skew matrix.

18. In order that previous work on the above matters may be known, I append hereto a first approximation to a complete list of writings on Matrices. A supplementary list, to which I invite contributions, will be published when it is sufficiently bulky to warrant attention.

LIST OF WRITINGS ON THE THEORY OF MATRICES.

(1857–1893.)

1857.

CAYLEY, A. A memoir on the theory of matrices. Phil. Trans. R. S. London, CXLVIII, pp. 17–37; or Collected Math. Papers, II, pp. 475–496.

CAYLEY, A. A memoir on the automorphic linear transformation of a bipartite quadric function. Phil. Trans. R. S. London, CXLVIII, pp. 39–46; or Collected Math. Papers, II, pp. 497–505.

1867.

LAGUERRE, Sur le calcul des systèmes linéaires. Journ. de l'École Polyt., XXV, pp. 215–264.

1870.

PEIRCE, BENJAMIN. Linear associative algebra. American Journ. of Math., IV, pp. 97–215.

PEIRCE, CHAS. S. Description of a notation for the logic of relatives. Mem. Amer. Acad. Sci., IX.

1872.

SPOTTISWOODE, W. Remarks on some recent generalizations of algebra. Proc. Lond. Math. Soc., IV, pp. 147–164.

1877.

FROBENIUS, Ueber lineare Substitutionen und bilineare Formen. Crelle's Journ., LXXXIV, pp. 1–63.

1879.

CAYLEY, A. On the matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and in connection therewith the function $\dfrac{ax+b}{cx+d}$. Messenger of Math., IX, pp. 104–109.

1882.

SYLVESTER, J. J. Sur les puissances et les racines de substitutions linéaires. Comptes Rendus ...., Paris, XCIV, pp. 55–59.

SYLVESTER, J. J. Sur les racines des matrices unitaires. Comptes Rendus ...., Paris, XCIV, pp. 396–399.

SYLVESTER, J. J. On the properties of a split matrix. Johns Hopkins Univ. Circular, pp. 210, 211.

1883.

BUCHHEIM, A. An identity in the theory of matrices. Messenger of Math., XIII, pp. 65, 66.

SYLVESTER, J. J. On the equation to the secular inequalities in the planetary theory. Phil. Mag. 5th ser. XVI, pp. 267–269.

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FORSYTH, A. R. Proof of a theorem by Cayley in regard to matrices. Messenger of Math., XIII, pp. 139–142.

1884.

SYLVESTER, J. J. Lectures on the principles of universal algebra. Amer. Journ. of Math., VI, pp. 270–286.

SYLVESTER, J. J. On the solution of a class of equations in quaternions. Phil. Mag. 5th ser. XVII, pp. 392–397.

SYLVESTER, J. J. Sur les équations monothétiques. Comptes Rendus ...., Paris, XCIX, pp. 13–15.

SYLVESTER, J. J. Sur l'équation en matrices $px=xq$. Comptes Rendus ...., Paris, pp. 67–71.

SYLVESTER, J. J. Sur l'équation en matrices $px=xq$. Comptes Rendus ...., Paris, XCIX, pp. 115, 116.

SYLVESTER, J. J. Sur la solution du cas le plus général des équations linéaires en quantités binaires, c'est-à-dire en quaternions ou en matrices du second ordre. Comptes Rendus ...., Paris, XCIX, pp. 117, 118.

SYLVESTER, J. J. (Theorem regarding two matrices which have a latent root in common.) Educational Times, XXXVII, pp. 297, 385; or Math. from Educ. Times, XLII, p. 101.

SYLVESTER, J. J. Sur la résolution générale de l'équation linéaire en matrices d'un ordre quelconque. Comptes Rendus ...., Paris, XCIX, pp. 409–412.

SYLVESTER, J. J. Sur la résolution générale de l'équation linéaire en matrices d'un ordre quelconque. Comptes Rendus, ... , Paris, XCIX, pp. 432–436.

SYLVESTER, J. J. Sur les deux méthodes, celle de Hamilton et celle de l'auteur, pour résoudre l'équation linéaire en quaternions. Comptes Rendus . . . . , Paris, XCIX, pp. 473–476.

SYLVESTER, J. J. Sur l'achèvement de la nouvelle méthode pour résoudre l'équation linéaire la plus générale en quaternions. Comptes Rendus . . . . , Paris, XCIX, pp. 502–505.

SYLVESTER, J. J. Sur l'équation linéaire trinôme en matrices d'un ordre quelconque. Comptes Rendus . . . . , Paris, XCIX, pp. 527–529.

BUCHHEIM, A. Proof of Sylvester's "Third Law of Motion." Phil. Mag. 5th ser. XVIII, pp. 459–460.

SYLVESTER, J. J. Sur la solution explicite de l'équation quadratique de Hamilton en quaternions ou en matrices du second ordre. Comptes Rendus . . . . , Paris, XCIX, pp. 555–558.

SYLVESTER, J. J. Sur les conditions de l'existence de racines égales dans l'équation du second degré de Hamilton, et sur une méthode générale pour résoudre une équation unilaterale de n'importe quel degré en matrices d'un ordre quelconque. Comptes Rendus . . . . , Paris, XCIX, pp. 621–631.

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CAYLEY, A. On the quaternion equation $qQ - Qq' = 0$. Messenger of Math., XIV, pp. 108–112.

SYLVESTER, J. J. On Hamilton's quadratic equation and the general unilateral equation on matrices. Phil. Mag. 5th ser. XVIII, pp. 454–458.

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